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Notes on the Deconfining Phase Transition

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Abstract

I review the deconfining phase transition in an $SU(N)$ gauge theory without quarks. After computing the interface tension between $Z(N)$ degenerate vacua deep in the deconfined phase, I follow Giovannangeli and Korthals Altes, and suggest a new model for (discrete) Polyakov loop spins. Effective theories for (continuous) Polyakov loop spins are constructed, including those with $Z(N)$ charge greater than one, and compared with Lattice data. About the deconfining transition, the expectation values of $Z(N)$ singlet fields ("quarkless baryons") may change markedly. Speculations include: a possible duality between Polyakov loop and ordinary spins in four dimensions, and how $Z(N)$ bubbles might be guaranteed to have positive pressure.

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1 Overview

In these lectures I review the deconfining phase transition in a "pure" $SU(N)$ gauge theory, without dynamical quarks. Gauge theories are ubiquitous in physics, so their phase transitions are manifestly of fundamental importance. Two examples may include the collisions of large nuclei at high energy and the early universe.

The phase transitions of gauge theories without quarks are of especial interest, since then the order parameter, and many other aspects of the phase transition, can be characterized precisely [1, 2, 3]. While of course QCD includes quarks, this is not an academic exercise. Recent results from the Lattice on "flavor independence" — for both the pressure [4, 5] and quark susceptibilities [6] — suggest that the results from the pure glue theory may be, in a surprising and unexpected fashion, relevant for QCD. (Whether flavor independence can be generalized when there are many light flavors is not known.)

Albeit indirectly, the Lattice has already told us much about what happens in a pure gauge theory with three colors. By asymptotic freedom, at infinite temperature the pressure is that for an ideal gas of gluons. In the confined phase, the pressure is very small, essentially zero. So the question is, as the pressure turns on at the transition temperature T_c, how rapidly does it approach the ideal gas limit? The Lattice tells us relatively quickly: by about $2T_c$, it is already 80% of the way to ideality. The "2" in $2T_c$ is meant schematically; it is certainly not, say, $10T_c$. Above $2T_c$, the pressure then approaches ideality slowly, from below.

This suggests that from temperatures of $2T_c$ on up, that the theory is some sort of quasiparticle gas of thermal quarks and gluons; *i.e.*, a Quark-Gluon Plasma. By this I mean that after suitable dressing from bare into quasi-particles, the residual interactions are weak. This is seen from the Lattice: like the ratio of the true to the ideal pressure, the ratio of (gauge-invariant) masses to the temperature also vary slowly above $2T_c$ [7, 8].

It is known that for the free energy, direct perturbative calculations fail at astronomically high temperatures [9]; there is only a perturbative Quark-Gluon Plasma above temperatures of $\sim 10^7$ GeV. Thus from temperatures of 10^7 GeV down to $2T_c$, the theory is what I call a non-perturbative Quark-Gluon Plasma.

One approach to the non-perturbative QGP is to fold in the effects of Debye screening. It is known that the pressure, as obtained from the Lattice, can be fit to an ideal gas of massive gluons all the way down to T_c [10]. The problem is that as introduced, these masses aren't gauge invariant. Further, in the end one is just fitting one function of temperature, the pressure, to another, thermal masses. (Still, it is most intriguing that these fit thermal masses become large near T_c.)

Another approach is provided by the resummation of Hard Thermal Loops (HTL) [11, 12, 13, 14]. For the gauge field, if A_0 is the time-like component of the vector potential, the Debye mass term is just $\sim \mathrm{tr}(A_0^2)$. For gluons, HTL's are the gauge invariant, analytic continuation of this mass term from imaginary to real time. At present, though, HTL resummation has been used mainly to compute the pressure. The crucial test, yet remaining, are the results which it gives for Polyakov loop correlation functions [7, 15].

A method to compute all static correlation functions in the non-perturbative Quark-Gluon Plasma starts with the (perturbative) construction of an effective theory in three dimensions [16]. From the original gauge theory at $T \neq 0$, static magnetic fields produce three dimensional gluons, while static electric fields give A_0, as an adjoint scalar coupled to these gluons. Due to the power-like infrared divergences of gauge theories in three dimensions, perturbative calculations are useless in this effective theory. Since the effective theory is purely bosonic, though, static correlation functions can be efficiently computed by numerical means on the Lattice. While heroic perturbative calculations are required, for three colors this method appears to work down to $2T_c$, where the method itself indicates its failure. To be fair, this method only yields static correlation functions, and not those in real time. With HTL resummation, this continuation is almost automatic.

What about below $2T_c$? If the transition is strongly first order, then presumably the quasiparticle regime extends all of the way down to T_c. Lattice data suggests that for four colors [17], the transition is strongly first order. (Although data on correlation lengths near T_c is absent.) For more than four colors, if the deconfining transition is like the Potts model, then it becomes more strongly first order as N increases. On the Lattice, "reduced" models have a first order transition at infinite N [18]. In the continuum, for really no good reason, I remain unconvinced [19].

For two colors, however, the deconfining transition is almost certainly of second order [20]. For three colors, the transition is of first order [21], as predicted by Svetitsky and Yaffe [3]. Recent results suggest, however, that the transition is so weakly first order that it is more accurate to speak of a "nearly second order" transition [22].

This is seen most clearly not from the pressure, but from the behavior of electric and magnetic masses. I define the electric, m_{el}, and magnetic masses, m_{mag}, in a gauge invariant way, from the fall-off of the two-point functions for Polyakov loops, and the (trace of the) magnetic field squared, respectively. In the perturbative QGP, $m_{el} > m_{mag}$ [7], while in the non-perturbative QGP, $m_{el} \sim m_{mag}$ [7, 8]. In contrast, in the transition regime m_{mag}/T is approximately constant, but m_{el}/T appears to drop by a factor of ten as $T : 2T_c \to T_c^+$ [22]. A similar drop in the string tension is seen as $T : 0 \to T_c^-$ [22].

The broad outlines of the appropriate effective theory in the transition region for a (nearly) second order deconfining transition have been known for some time [1, 2, 3, 23, 24]. In the QGP regime, one deals with A_0. In the transition region, one trades A_0 in for the thermal Wilson line. Traces of powers of the thermal Wilson line give Polyakov loops, which are gauge invariant.

The effective theory of Polyakov loops is just beginning [25, 26, 27, 28, 29, 30, 31]. If true, instead of the A_0 quasiparticles of the QGP, near T_c it is more useful to view the theory as a *condensate* of Polyakov loops. This could produce dramatic signatures in heavy ion collisions, with hadronization at T_c computed semiclassically from the decay of Polyakov loop condensates [26, 27, 28].

In these Lectures I provide some background to understand these questions. After explaining why $SU(N)$ gauge theories have $Z(N)$ degenerate vacua [2], I compute the interface tension between these vacua at high temperature [32, 33, 34, 35, 36, 37, 38, 39]. Viewing the $Z(N)$ vacua as discrete spins, and using results of Giovannangeli and Korthals-Altes [37], I propose a new spin model, distinct from the usual Potts model. (The order of the transition seems to agree with Potts for $N \leq 4$, but is unknown for $N > 4$.) I then develop an effective theory of Polyakov loop spins, considered as continuous variables. Even with the limited amount of relevant Lattice data which exists at present, the form of this effective theory is sharply constrained. Especially intriguing is the possibility that the expectation values of fields which are singlets under the $Z(N)$ symmetry — which I term quarkless baryons — change suddenly about T_c.

The lectures are in part pedagogical, in part base speculation. The latter includes a possible duality in four dimensions, and how $Z(N)$ bubbles, which in perturbation theory can have negative pressure [39], might be ensured of positive pressure non-perturbatively. I also review what is known about renormalization of Wilson loops and Wilson lines [40, 41, 42, 43].

In these lectures I only discuss the Polyakov Loops Model [25, 26, 27, 28, 29, 30, 31] in passing. I hope to provide a basis for understanding its motivation. At present, a major unsolved problem is the analytic continuation from imaginary to real time. In abelian gauge theories, the analytic continuation of the Debye mass term gives the Random Phase Approximation [44]; in nonabelian theories, it gives Hard Thermal Loops [11]. Absent any results whatsoever, I do have the temerity to coin a phrase for the analytic continuation of the Polyakov Loops Model to real time, as the "*Nonabelian Random Phase Approximation*". Time (dependence) will tell.

1.1 Saturation

My motivation for studying this subject is its relevance for the collisions of large nuclei at very high energies. Thus at the outset, I wish to add some general comments about nucleus-nucleus collisions, and especially how it might relate to models of saturation.

Consider a completely implausible situation, the collision of two neutron stars (say) at relativistic energies. For a neutron star, the transverse area is essentially infinite on nuclear scales. If the stars completely overlap (zero impact parameter), then at very high energies, a nearly baryon-free region is generated between them (about zero rapidity). The system starts out with energy density, but no pressure. It is then reasonable to think that the system builds up pressure, and thermalizes, before it flies apart.

The crucial question for the collisions of two large nuclei is whether for gold or lead nuclei, with $A \sim 200$, that this finite value of A is close to infinity, or represents some intermediate regime. This will be decided by experiment, at the SPS, RHIC, and the LHC. After one year of running, it appears clear that *something* dramatic has happened between SPS and RHIC energies [45]. Precisely what is still being sorted out.

I wish to comment here on the relevance of "saturation" at these energies [46]. At a very pedestrian level, this can be viewed as a type of finite size effect at $A < \infty$. Consider a collision in the rest frame of one nuclei. The diameter of the other nucleus, with $A \sim 200$, is ~ 15 fm. This distance becomes Lorentz contracted. Thus we can ask, in the rest frame of one nuclei, when does the incident nuclei look like a pancake of negligible width? We want the distance to be really small on typical hadronic scales. If a typical hadronic scale is 1 fm, then, a small scale might be $1/4 \to 1/3$ fm, say. To contract 15 fm down to these sizes then requires a center of mass energy per nucleon pair, $\sqrt{s}/A$, on the order of $\sim (3 \to 4) \times 15 = 45 \to 60$ GeV. While an extremely naive estimate, this does seem to be a reasonable estimate for the $\sqrt{s}/A$ where the distribution of particles in AA collisions changes dramatically, developing a "central plateau" as a function of pseudo-rapidity. Thus perhaps the appearance of the central plateau is where the effects of saturation first appear.

The details are far more involved, but but our understanding of saturation, which is known formally as the Color Glass [46], gives us some confidence that the system is described in terms of "saturated" gluons, with a characteristic momentum $p_{sat} \sim 1 - 2$ GeV.

The Color Glass changes all assumptions in one fundamental respect. Following Bjorken, the usual assumption is that the system thermalizes on a "typical" hadronic time scale, ~ 1 fm/c. If saturation kicks in, however, all typical scales are then given in terms of $1/p_{sat}$, which is a *much* smaller time scale, $\sim .1 - .2$ fm/c. If this is the relevant scale, then evolution to a thermal state, for a nucleus of size 6 fm, appears much more plausible. Certainly it yields testable predictions, which are testable experimentally.

2 $Z(N)$ symmetries in $SU(N)$

I start by reviewing how, following 't Hooft [2], a global $Z(N)$ symmetry emerges from a local $SU(N)$ gauge theory. The action, including quarks, is

$$\mathcal{L} = \frac{1}{2}\, \mathrm{tr}\, G^2_{\mu\nu} \ + \ \overline{q}\, i \, \not{D}\, q \ , \tag{1}$$

where

$$D_\mu = \partial_\mu - igA_\mu \quad , \quad G_{\mu\nu} \ = \ \frac{1}{-ig}\,[D_\mu, D_\nu] \ ; \tag{2}$$

$A_\mu = A^a_\mu t^a$, with the generators of $SU(N)$ normalized as $\mathrm{tr}(t^a t^b) = \delta^{ab}/2$. The Lagrangian is invariant under $SU(N)$ gauge transformations Ω,

$$D_\mu \to \Omega^\dagger D_\mu \Omega \quad , \quad q \to \Omega^\dagger q \ . \tag{3}$$

As an element of $SU(N)$, Ω satisfies

$$\Omega^\dagger \Omega = \mathbf{1} \ , \ \det \Omega = 1 \ . \tag{4}$$

Here Ω, as a local gauge transformation, is a function of space-time.

There is one especially simple gauge transformation — a constant phase times the unit matrix:

$$\Omega_c \ = \ e^{i\phi}\, \mathbf{1} \ . \tag{5}$$

To be an element of $SU(N)$, the determinant must be one, which requires

$$\phi = \frac{2\pi j}{N} \quad , \quad j = 0, 1 \ldots (N-1) \ . \tag{6}$$

Since an integer cannot change continuously from point to point, this defines a global $Z(N)$ symmetry.

2.1 $Z(N)$ at nonzero temperature

As a particular gauge transformation, $Z(N)$ rotations are always a symmetry of the Lagrangian, either with or without quarks. I now show that with quarks, they are not a symmetry of the theory, because they violate the requisite boundary conditions.

I work in Euclidean space-time at a temperature T, so the imaginary time coordinate τ, is of finite extent, $\tau : 0 \to \beta = 1/T$. The proper boundary conditions in imaginary time are dictated by the quantum statistics which the fields must satisfy. As bosons, gluons must be periodic in τ; as fermions, quarks must be anti-periodic:

$$A_\mu(\vec{x},\beta) \;=\; +A_\mu(\vec{x},0) \quad , \quad q(\vec{x},\beta) \;=\; -q(\vec{x},0) \; . \tag{7}$$

Obviously any gauge transformation which is periodic in τ respects these boundary conditions. 't Hooft noticed, however, that one can consider more general gauge transformations which are only periodic up to Ω_c:

$$\Omega(\vec{x},\beta) = \Omega_c \quad , \quad \Omega(\vec{x},0) = 1 \; . \tag{8}$$

Color adjoint fields are invariant under this transformation, while those in the fundamental representation are not:

$$A^\Omega(\vec{x},\beta) \;=\; \Omega_c^\dagger A_\mu(\vec{x},\beta)\Omega_c \;=\; A_\mu(\vec{x},\beta) \;=\; +\,A_\mu(\vec{x},0) \; , \tag{9}$$

$$q^\Omega(\vec{x},\beta) \;=\; \Omega_c^\dagger\, q(\vec{x},\beta) \;=\; e^{-i\phi} q(\vec{x},\beta) \;\neq -\,q(\vec{x},0) \; . \tag{10}$$

Here I have used the fact that Ω_c, as a constant phase times the unit matrix, commutes with any $SU(N)$ matrix. Consequently, pure $SU(N)$ gauge theories have a global $Z(N)$ symmetry which is spoiled by the addition of dynamical quarks.

In the pure glue theory, an order parameter for the $Z(N)$ symmetry is constructed using the thermal Wilson line:

$$\mathbf{L}(\vec{x}) = \mathbf{P}\exp\left(ig \int_0^\beta A_0(\vec{x},\tau)d\tau \right) \; ; \tag{11}$$

g is the gauge coupling constant, and A_0 the vector potential in the time direction. The symbol $\mathbf{P}$ denotes path ordering, so that the thermal Wilson line transforms like an adjoint field under local $SU(N)$ gauge transformations:

$$\mathbf{L}(\vec{x}) \to \Omega^\dagger(\vec{x},\beta)\, \mathbf{L}(\vec{x})\, \Omega^\dagger(\vec{x},0) \; . \tag{12}$$

The Polyakov loop [1] is the trace of the thermal Wilson line, and is then gauge invariant:

$$\ell = \frac{1}{N}\, \mathrm{tr}\, \mathbf{L} \; . \tag{13}$$

Under a global $Z(N)$ transformation, the Polyakov loop ℓ_1 transforms as a field with charge one:

$$\ell \to e^{i\phi} \ell \; . \tag{14}$$

At very high temperature, the theory is nearly ideal, so $g \approx 0$, and naively one expects that $\langle \ell \rangle \sim 1$. Instead, the allowed vacua exhibit a N-fold degeneracy:

$$\langle \ell \rangle \; = \; \exp\left(\frac{2\pi i j}{N}\right) \, \ell_0 \quad , \quad j \; = \; 0, 1 \ldots (N-1) \; , \tag{15}$$

defining ℓ_0 to be real; $\ell_0 \to 1$ as $T \to \infty$. Any value of j is equally good, and signals the spontaneous breakdown of the global $Z(N)$ symmetry.

At zero temperature, confinement implies that ℓ_0 vanishes [2]. The modulus, ℓ_0, is nonzero above T_c:

$$\ell_0 \; = \; 0 \quad , \quad T < T_c \quad ; \quad \ell_0 \; > \; 0 \quad , \quad T > T_c \; . \tag{16}$$

As is standard, if ℓ_0 turns on continuously at T_c, the transition is of second order; if it jumps at T_c, it is of first order. What is atypical is that the $Z(N)$ symmetry is broken at high, instead of low, temperatures. For a heuristic explanation of this in terms of $Z(N)$ spins, see sec. (3.2).

One can also understand what it means to say that the global $Z(N)$ symmetry is violated by the presence of dynamical quarks. In the pure glue theory, $Z(N)$ rotations take us from one degenerate vacua to another, all of which have the same pressure; see sec. (3.2). Adding dynamical quarks (with real masses), the stable vacuum is that for which $\langle \ell \rangle$ is real, $j = 0$ [23, 24]. As discussed in sec. (3.4), because of quarks the pressure for a $Z(N)$ state with $j \neq 0$ is less than the stable vacuum; thermodynamically, they are unstable. What is exciting to some of us [38], however, is the possibility that these $Z(N)$ rotated states might be *meta*stable. Such "$Z(N)$ bubbles" could have cosmologically interesting consequences [38]. Others contest whether any of this makes any sense [39].

The usual interpretation of the Polyakov loop is as the free energy of an infinitely heavy test quark [47]:

$$\langle \ell \rangle = \exp\left(-F_{test}/T\right) \; . \tag{17}$$

This cannot be quite right: when $N = 2$, the left hand side can be of either sign, while for $N \geq 3$, it is complex. In contrast, free energies are real, so the right hand side is positive.

I suggest a different view. Consider the propagator for a scalar in a background gauge field (the extension to fermions is irrelevant here). This propagator is given by a Feynman sum over paths:

$$\Delta = \int \mathcal{D}x^\mu \exp\left(-\int ds \left(\frac{\dot{x}^2}{2} \; + \; m \; + \; igA^\mu \dot{x}^\mu\right)\right) \; , \tag{18}$$

with s the length of the path for the worldline of the particle, and $\dot{x} = dx^\mu/ds$. A very heavy quark moves in a straight line; in imaginary time, it sits wherever you put it. As a colored field, however, it also carries a color Aharonov-Bohm phase. This phase is nontrivial, and is precisely the thermal Wilson line. Thus: *the Polyakov loop, ℓ, is the trace of the propagator for a test quark.*

Confinement then means that (the trace of) this propagator vanishes. For two colors, for example, the confining vacuum is $Z(2)$ symmetric: it is composed of domains, of definite size, in which $\ell = +1$ and $\ell = -1$. As the test quark travels through each domain, it picks up one phase or the other. Over a very long path, these phases cancel out, giving zero overall. The same holds for higher N, except that there are then N types of domains. This picture appears analogous to the localization of an electron in a random potential [48].

$Z(N)$ rotations can be expressed in terms of the canonical formalism [2, 35]. In $A_0 = 0$ gauge, usually the partition function is strictly a trace of $\exp(-\mathcal{H}/T)$ ($\mathcal{H}$ is the corresponding Hamiltonian), sandwiched between the same state:

$$Z = \Sigma\ \langle\psi|\exp(-\mathcal{H}/T)|\psi\rangle. \tag{19}$$

The sum is over all gauge invariant states in the theory. In a canonical formalism, one inserts a projector to ensure that the states are gauge invariant, satisfying Gauss' Law, although the process is standard. Instead, what enters here is a "twisted" trace:

$$Z(\Omega_c) = \Sigma\ \langle\psi^{\Omega_c}|\exp(-\mathcal{H}/T)|\psi\rangle. \tag{20}$$

Here, ψ^{Ω_c} represents the gauge transform of ψ by the gauge transformation Ω_c. This is not automatically equal to ψ, because Gauss' Law only ensures that the state is invariant under local gauge transformations, and does not restrict its behavior under the global gauge transformations. This twisted trace is only possible in a gauge theory.

3 $Z(N)$ Vacua and Bubbles

3.1 Tunneling between Degenerate $Z(N)$ Vacua

Typically, discussions of gauge theories at nonzero temperature work up from zero temperature. But the zero temperature theory confines, which is complicated. Instead, I work down from infinite temperature, in a gas of nearly ideal gluons. I now compute the amplitude to tunnel from one $Z(N)$ vacua to another by semi-classical means [32, 33, 34, 37].

For simplicity, I work with two colors. To compute the interface tension, I put the system in a box:

$$\tau : 0 \to \beta \quad , \quad x, y : 0 \to L_t \quad , \quad z : 0 \to L\ . \tag{21}$$

I choose one arbitary direction, say the z-direction, and make that much longer than the transverse spatial directions, x and y, and than the direction in imaginary time, τ. I impose boundary conditions such that the Polyakov loop has one value in $Z(2)$ at one end of the box, and the other value at the other end of the box:

$$\ell(0) = 1 \quad , \quad \ell(L) = -1 \; . \tag{22}$$

With these boundary conditions, the system is forced to form an interface between the two ends of the box. The simplest interface is one which is constant in the transverse directions. Thus the natural expectation is that with the above boundary conditions, the action is

$$S_{inter} \; = \; L_t^2 \frac{c}{g^2} \quad , \quad c \; = \; c_0 \; + \; g c_1 \; + \ldots \tag{23}$$

One expects the result to start as $\sim 1/g^2$, as a semiclassical probability for tunneling in weak coupling. This is standard with instantons, *etc.* Higher order corrections to the leading term, c_0, are generated by including quantum effects, and produce the corrections c_1, *etc.*

As an ansatze for the $Z(N)$ interface, I take

$$A_0^{cl}(z) \; = \; \frac{\pi T}{g} \, q(z) \, \sigma_3 \quad , \quad \sigma_3 \; = \; \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \tag{24}$$

With this ansatz, the Polyakov loop is

$$\ell(z) \; = \; \cos\left(\pi q(z)\right) \; . \tag{25}$$

Thus the boundary conditions are satisfied by taking

$$q(0) \; = \; 0 \quad , \quad q(L) \; = \; 1 \; . \tag{26}$$

At the classical level, the action of the above configuration is:

$$\mathcal{S}^{cl} \; = \; \int_0^\beta d\tau \int d^3x \, \frac{1}{2} \, \mathrm{tr} \, \left(\left(G_{\mu\nu}^{cl} \right)^2 \right)$$
$$= \; L_t^2 \, \frac{2\pi^2 T}{g^2} \int dz \left(\frac{dq}{dz} \right)^2 \; . \tag{27}$$

Unsurprisingly, the action for the gauge field becomes a kinetic term for the classical field. There is only a kinetic term, since the classical field commutes with itself.

This implies, however, that at the classical level, there is *no* difference between the two vacua, or indeed any state with $q \neq 0$! One can take $q(z) = z/L$; then the action is $\sim L/L^2 \sim 1/L$, and vanishes as $L \to \infty$. In terms of the above,

$$c_0 \; = \; 0 \; . \tag{28}$$

I now show that quantum corrections generate a nonzero value for c_1. Since c_0 vanishes, this is then the leading term in a semiclassical expansion; the tunneling probability is then not $\sim 1/g^2$, but only $\sim 1/g$.

To show this, it is necessary to compute the quantum corrections about the above semiclassical configuration, taking

$$A_\mu \; = \; A_\mu^{cl} \; + \; A_\mu^q \; . \tag{29}$$

The computation is a bit involved, but an excellent exercise in the use of the background field method [50], which is always good to know.

It is convenient to take background field gauge,

$$D_\mu^{cl} A_\mu^q \; = \; \partial_\mu A_\mu^q \; - \; ig[A_\mu^{cl}, A_\mu^q] \; = \; 0 \; . \tag{30}$$

I work in euclidean space-time with $(++++)$ metric. Note that the appropriate covariant derivative is that in the adjoint representation. With this gauge fixing, the Lagriangian density is

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}\left(G^{cl}\right)^2 \; + \; \frac{1}{\xi}\,\mathrm{tr}\left(D^{cl}\cdot A^q\right)^2 + \overline{\eta}\left(-D^{cl}\cdot D\right)\eta \; , \tag{31}$$

suppressing ugly vector indices.

With this form, it is easy integrating out the quantum fields to one loop order, and obtain the quantum action

$$\mathcal{S}^q(A^{cl}) \; = \; \frac{1}{2}\,\mathrm{tr}\log\left(\Delta_{\mu\nu}^{-1}\right) \; - \; \mathrm{tr}\log\left(\Delta_\eta^{-1}\right) \; . \tag{32}$$

At one loop order, the full effective action is the sum of the classical action, (27), and the quantum action, (32). The quantum action involves the inverse propagators in a background field. That for the gluon is

$$\Delta_{\mu\nu}^{-1} = \; -D_{cl}^2\delta_{\mu\nu} \; + \; (1-\xi^{-1})D_\mu^{cl}D_\nu^{cl} \; + \; 2ig\,[G_{\mu\nu}^{cl}, \;\;] \; , \tag{33}$$

while that for the ghost is (to lowest order in g)

$$\Delta_\eta^{-1} \; = \; -D_{cl}^2 \; . \tag{34}$$

These results are valid for an arbitrary background gauge field.

I now make a crucial assumption, and assume that the field $q(z)$ is constant in space. For the relevant tunneling amplitude, in fact $q(z)$ does depend upon z; what happens, however, is that for the quantum action, this variation only enters to higher order in the coupling constant.

This assumption vastly simplifies the problem. Since A_0 lies in the σ_3 direction, it is a diagonal matrix, and covariant derivatives commute:

$$[D_\mu^{cl}, D_\nu^{cl}] \; \sim \; G_{\mu\nu}^{cl} \; = \; 0 \; . \tag{35}$$

For example, one can easily show that the quantum action is independent of the gauge fixing condition. The variation of the quantum action with respect to the gauge fixing parameter ξ is

$$\frac{\partial}{\partial \xi^{-1}} \, \mathcal{S}^q \; = \; \frac{1}{2} \, \mathrm{tr} \left(-D^{cl}_\mu D^{cl}_\nu \Delta^{cl}_{\mu\nu} \right) \; , \tag{36}$$

with $\Delta^{cl}_{\mu\nu}$ the gluon propagator. Normally, this is difficult to compute. In the present example, however, if covariant derivatives can be assumed to commute with each other, then they can be treated just like ordinary derivates, so that

$$\Delta^{cl}_{\mu\nu} \; = \; \frac{\delta^{\mu\nu}}{-D^2_{cl}} \; + \; (1-\xi) \frac{D^\mu_{cl} D^\nu_{cl}}{(-D^2_{cl})^2} \; . \tag{37}$$

Consequently,

$$\frac{\partial}{\partial \xi^{-1}} \, \mathcal{S}^q \; = \; \frac{\xi}{2} \, \mathrm{tr}(1) \; . \tag{38}$$

Thus there is gauge dependence in the quantum action, but it is completely independent of the background field, and so can be safely ignored.

Consequently, I take background Feynman gauge, $\xi = 1$, and

$$\mathcal{S}^q \; = \; \mathrm{tr} \, \log \left(-D^2_{cl} \right) \; . \tag{39}$$

The overall factor is expected for a massless gauge field, with two (spin) degrees of freedom.

To compute the determinant in this background field, I introduce the "ladder" basis,

$$\sigma^+ \; = \; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \sigma^- \; = \; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; . \tag{40}$$

This is useful because of the commutation relations:

$$[\sigma_3, \sigma^\pm] \; = \; \pm \, 2\sigma^\pm \; , \tag{41}$$

so that the covariant derivative becomes

$$D^{cl}_0 \sigma^\mp \; = \; \left(\partial_0 - ig \left(\frac{\pi T}{g} q \right) [\sigma_3, \;] \right) \sigma^\mp \; = \; i \, (2\pi T) \, (n \pm q) \sigma^\mp \; . \tag{42}$$

In the last expression, I have gone to momentum space. Remember that given the periodic boundary conditions at nonzero temperature,

$$k_0 \; = \; i \, 2\pi T \, n \; , \tag{43}$$

where for a bosonic field, such as a gluon, the periodic boundary conditions require that n be an integer, $n = 0, \pm 1, \pm 2 \ldots$. In the present case, it is handy to introduce the shifted momentum,

$$k_0^\pm \; = \; i \, 2\pi T \, (n \pm q) \; . \tag{44}$$

In the trace, the sign of q doesn't matter, so that in momentum space,

$$\mathcal{S}^q \;=\; 2 \;\mathrm{tr}\; \log\left((k_0^+)^2 + \vec{k}^2\right) \;. \tag{45}$$

In computing integrals at nonzero temperature, the usual approach is to do the sum over the k_0's first by contour integration or the like, and then integrate over the spatial momentum. In the present example, instead it is better to first integrate over the spatial momentum, and then sum over the k_0's, using zeta functions tricks.

Only the variation of the action with respect to q,

$$\frac{\partial}{\partial q}\mathcal{S}^q \;=\; 8\pi T\,(\beta L_t^2 L)\; T \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3}\, \frac{k_0^+}{(k_0^+)^2 + \vec{k}^2} \;, \tag{46}$$

is needed. The integral over $\vec{k}$ can be done using dimensional regularization, viewing k_0^+ like a mass. Using the standard integral,

$$\int d^n k \, \frac{1}{(k^2 + m^2)^a} \;=\; \frac{\Gamma(a - n/2)}{\Gamma(a)}\, \frac{\pi^{n/2}}{(m^2)^{a-n/2}} \;, \tag{47}$$

the result is finite:

$$\begin{aligned} \frac{\partial}{\partial q}\mathcal{S}^q \;&=\; 8\pi L_t^2 L\; T \sum_{n=-\infty}^{+\infty} \; k_0^+ \left(\frac{-1}{4\pi}|k_0^+|\right) \\ &=\; -8\pi^2 T^3 L_t^2 L \sum_{n=-\infty}^{+\infty} \; (n+q)|n+q| \;. \end{aligned} \tag{48}$$

The sum over n, where n runs from minus infinity to plus infinity, is turned into a sum from zero to plus infinity:

$$= -8\pi^2 T^3 L_t^2 L \sum_{n=0}^{+\infty} \left((n+q)^2 - (n + (1-q))^2\right) \tag{49}$$

While these sums are very divergent, mathematicians know how to handle them. They are defined by the analytic continuation of the Riemann zeta-function:

$$\zeta(z, q) \;=\; \sum_{n=0}^{+\infty} \frac{1}{(n+q)^z} \;. \tag{50}$$

Using

$$\zeta(-2, q) \;=\; -\,\frac{1}{12}\,\frac{d}{dq}\left(q^2(1-q)^2\right) \;, \tag{51}$$

gives

$$\mathcal{S}^q \;=\; L_t^2 \, \frac{4\pi^2 T^3}{3} \, \int dz \; q^2 \, (1-q)^2 \; . \tag{52}$$

This expression is only valid for $q : 0 \rightarrow 1$; given the derivation, it is a periodic function in q with period one. Also note that in anticipation of later results, I have replaced a factor of the length L by $\int dz$.

Physically, the computation of quantum corrections has lifted the degeneracy in q by generating a potential for q. In the full effective action, it helps to rescale the length in the z direction, introducing

$$z' \;=\; \sqrt{\frac{2}{3}} \, gT \; z \; . \tag{53}$$

With this rescaling, the complete effective action at one-loop order is

$$\mathcal{S}^{eff} \;=\; \mathcal{S}^{cl} \;+\; \mathcal{S}^q \;=\; L_t^2 \; \frac{4\pi^2}{\sqrt{6}} \frac{T^2}{g} \; \mathcal{S}' \; , \tag{54}$$

where

$$\mathcal{S}' \;=\; \int \; dz' \; \left(\left(\frac{dq}{dz'} \right)^2 \;+\; q^2 \, (1-q)^2 \right) \; . \tag{55}$$

The factor of T^2 follows on dimensional grounds, as at high temperature, T is the only natural mass scale in the problem. (The renormalization mass scale doesn't appear until next to leading order [32, 33].)

What is most interesting is that the $1/g^2$, which we had expected, becomes only a $1/g$. This is because our effective action only acquires a potential at one loop order.

This begs the important question, why should this effective action be trusted to one loop order? What about the effects to higher loop order? The point is that in the new effective action, the relevant distance scale is not just $1/T$, but $1/(gT)$: notice the factor of gT in the definition of the rescaled length z'. Thus for small g, any variations in the effective action occur over much larger distance scales than $1/T$. This is why our method of derivation — ignoring the variation of $q(z)$ in the quantum action —- works. It does vary in space, but in weak coupling, this variation is very slow, and can be ignored.

Having reduced the effective action to the above form, we merely want the "kink" which interpolates between the two vacua. While the general form of the kink is well known, in fact we only need the action. To compute the action, we note that the "energy" ϵ for this system is conserved:

$$\epsilon \;=\; \left(\frac{dq}{dz'} \right)^2 - q^2 (1-q)^2 \; . \tag{56}$$

Here we view the spatial coordinate z' as a kind of time; saying the energy is conserved means that it is independent of z'. The energy is then a constant of motion; with these boundary conditions, this energy vanishes. Zero energy implies

$$\frac{dq}{dz'} \;=\; q(1-q)\;. \tag{57}$$

Using this,

$$\mathcal{S}' \;=\; 2\int dz'\, q^2(1-q)^2 \;=\; 2\int_0^1 dq\, q(1-q) \;=\; \frac{1}{3}\;. \tag{58}$$

Putting everything together,

$$\mathcal{S}^{eff} \;=\; L_t^2\,\frac{4\pi^2}{3\sqrt{6}}\,\frac{T^2}{g}\;, \tag{59}$$

which is the final result for two colors.

This demonstrates that $c_1 \neq 0$; the interface tension vanishes classically, but is generated through quantum effects. The most interesting feature of the analysis is how the tunneling amplitude goes from the expected $1/g^2$ to just $1/g$. There are examples in string theory where tunneling amplitudes are not $1/g^2$, but only $1/g$. These examples appear special to string theory, as the appearance of the coupling constant in this fashion is more or less natural. That is not the case here; the $1/g$ is really novel. The appearance of the inverse distance scale gT is reasonable, as the Debye screening mass in a thermal bath.

3.2 Spins and Polyakov Loop Spins: Duality?

Remember the behavior of a usual Ising magnet, in which spins $\sigma_i = \pm$ interact through a coupling constant J_{mag}. The Hamiltonian is:

$$\mathcal{H} \;=\; -\,J_{mag}\,\Sigma_{i,\hat{n}}\;\sigma_i\cdot\sigma_{i+\hat{n}}\;. \tag{60}$$

The sum is over all lattice sites, i, and nearest neighbors to i, $\hat{n}$. The spins align at low temperatures, and disorder at high temperatures. This is just because the partition function is $\mathcal{Z} \sim \exp(-\mathcal{H}/T)$. In magnets, the spin-spin coupling is more or less independent of temperature, so that with $\sim J_{mag}/T$ in the exponential, ordering wins at low temperature, and loses at high temperature.

The interface tension can easily be estimated. The simplest interface is to take all spins on the left hand side spin up, and all on the right hand side, spin down. If a is the lattice spacing, then the interface tension, defined as above, is $\sim J$; by construction, its width is a. This very sharp interface is not the configuration of lowest energy, but the true interface tension is of order $\sim J$, with a width of order, a. Another example, more familiar to field theorists, is given by a scalar field with a double well potential [36].

Now consider an effective lagrangian for Polyakov loops. Over distances $> 1/T$, the four-dimensional theory reduces to an effective theory of spins in three spatial dimensions. For two colors, Polyakov loops are a type of $Z(2)$ spin, with $\ell = \pm$. From the above $\mathcal{S}^{eff}$,

$$J_{Polyakov} \sim \frac{T^2}{g} \; . \tag{61}$$

Now it is easy to understand why Polyakov loop spins order at high, instead of low, temperature. For magnets, the partition function involves $\exp(-J_{mag}/T)$; for Polyakov loop spins, instead we have $\exp(-J_{Polyakov}/T)$; but as $J_{Polyakov} \sim T^2$, in all the temperature dependence in the exponential is not $1/T$, as for ordinary magnets, but T!

This also leads to a natural conjecture of duality: that the temperature for Polyakov loop spins, and an ordinary magnet, are related as

$$T_{Polyakov} \sim \frac{1}{T_{mag}} \; . \tag{62}$$

I have assumed that the variation of the gauge coupling constant with temperature can be neglected.

This argument is extremely heuristic, and carries an important qualification. For ordinary spins, the lattice spacing is of course fixed. (This is true as well for a scalar field with double well potential [36].) From the derivation of the interface tension above, however, the width of the interface is the inverse Debye mass, $\sim 1/(gT)$. Thus the argument fails in the limit of high temperature, since then the size of any single domain is becoming very large, $\sim \sqrt{\log(T)}/T$, as $T \to \infty$. This doesn't contradict the conclusion of ordering at high temperature, since any single domain is, by definition, an ordered state.

Now assume that the transition is of second order, as happens for two colors [20]. Then the correlation length diverges at T_c; with Polyakov loop spins, it decreases as T increases from T_c. This divergence is determined as usual by scaling at a critical point. Even so, the underlying length scale which fixes the lattice spacing for effective Polyakov loop spins is fixed, set by a mass scale proportional to T_c, *etc.* Near T_c, we have implicitly made the assumption that the interface tension remains proportional to $\sim T^2$. This cannot be true very near T_c, since for a second order transition, the interface tension must vanish at T_c.

So is the argument of any use? Well, if d is the number of space-time dimensions, then simply on geometric grounds, $J_{Polyakov} \sim T^{d-2}$. In three dimensions, then, $J_{Polyakov} \sim T$; depending upon the value of $J_{Polyakov}/T$, the system can still order above T_c, so there is no obvious contradiction. However, it does suggest that the *width* of the critical region is much *narrower* in four, as opposed to three, dimensions. For $SU(2)$ gauge theories, this comparison is of interest in its own right.

For a strongly first order transition, at first sight one might think that one should be able to directly check if $J_{Polyakov} \sim T^{d-2}$. This is complicated by the fact that near T_c, $Z(N)$ states don't tunnel directly from one to another, but from one $Z(N)$ state, to the symmetric vacuum, to another $Z(N)$ state.

3.3 Polyakov Loop Spins: Potts versus GKA

The above analysis can be extended to more than two colors. For two colors, there is only one interface tension, between $\ell = +1$ and $\ell = -1$. For N colors, the vacuum is one of the Nth roots of unity, $\ell = \exp(2\pi i j/N)$, $j = 1 \ldots (N-1)$. By charge conjugation, under which $\ell \to \ell^*$, the states j and $N-j$ are equivalent. There are then about $\sim N/2$ distinct interface tensions.

At any N, the smallest interface tension is between $j = 0$ and $j = 1$. Defining $\mathcal{S}^{eff} = \alpha_1 L_t^2$, then at next to leading order,

$$\alpha_1 \;=\; \frac{4(N-1)\pi^2}{3\sqrt{3N}} \, \frac{T^2}{g(T)} \left(1 - 12.9954... \, \frac{g^2\, N}{(4\pi)^2} + \ldots \right) , \tag{63}$$

where the running coupling constant $g^2(T)$ is defined using a modified $\overline{MS}$ scheme [37].

I remark that Boorstein and Kutasov [34] argued that due to infrared divergences, σ_1 is not $\sim 1/g$, but one over the magnetic mass scale, $\sigma_1 \sim 1/g^2$. While hardly conclusive, at least at next to leading order, there are no sign of infrared divergences.

The interface tension from $j = 0$ to $j = k$ has been computed by Giovannangeli and Korthals Altes (GKA) [37]. The result is amazingly simple:

$$\alpha_k \;=\; \frac{k\,(N-k)}{N-1}\, \alpha_1 \; . \tag{64}$$

Now I construct an effective theory of discrete $Z(N)$ spins. I forget about the factors of temperature which preoccupied me in the previous subsection; all that I am concerned with is the dependence on the distance between the $Z(N)$ spins. This suggests what I term the GKA model. The spins at each site of the lattice are integers j, $j = 0 \ldots (N-1)$; the Hamiltonian is

$$\mathcal{H}_{GKA} \;=\; J_{GKA}\, \Sigma_{j_i}\, k\,(N-k) \quad , \quad k = |j_i - j_{i+\hat{n}}|_{mod\ N} \; . \tag{65}$$

Tracing through the factors of N, and holding $g^2 N$ fixed as $N \to \infty$, the coupling constant $J_{GKA} > 0$ is of order one as $N \to \infty$.

The GKA model is in contrast to the Potts model, with Hamiltonian

$$\mathcal{H}_{Potts} \;=\; J\, \Sigma_{j_i}\, \delta_{k0} \; . \tag{66}$$

For the Potts model with $J > 0$, the energy is lowered if if two spins are equal, while if they differ — no matter by how much — the energy vanishes.

For two and three colors, there is no difference between the GKA model and the Potts model. For example, consider the case of three colors. Then $j = 1$ is equivalent to $j = 2$, so there is only one interface. That is, for the three roots of unity, any root is right next to the other two.

The Potts model is known to be of first order for any number of states greater than, or equal to, three. For the GKA model, in mean field theory the transition is of first order for four colors [37]; after all, the interaction between $j = 0$ and $j = 2$ is $4/3$ that between $j = 0$ and $j = 1$. Thus it would be very surprising if the GKA model wasn't also of first order when $N = 4$. Further, Lattice simulations of $SU(4)$ find a first order deconfining transition [17].

As the number of colors increases, though, the Potts and GKA models become increasingly different. Whatever N is, in the Potts model any spin state interacts equally strongly with any other spin state. In the GKA model, at large N spins only interact significantly with those which are close in spin space. For example, $\sigma_1 \sim N$, while $\sigma_j \sim N^2$ for $j \sim N/2$. It would be interesting to know the order of the phase transition in the GKA model at large N, both in mean field theory and numerically.

This assumes that the interaction between Polyakov loop spins — computed in the limit of very high temperature — remains the same all of the way down to T_c. This certainly is wrong for a second order transition, but the question here is if it is first order. Thus: does the interface tension stay $\sim j(N-j)$ (higher powers of $\sim j^2(N-j)^2$ only make the more less like Potts), or become constant, independent of j?

3.4 $Z(N)$ Bubbles

From the one-loop effective action, we can define a "potential" for q due to gluons, $\mathcal{V}_{gl}$. I will be schematic, suppressing all inessential details, and taking two colors for now. From the computation of the one-loop effective action, it is clear that the assumption of $L_t \ll L$ was actually a matter of words; one obtains identically the same result in an infinite volume. Thus from $\mathcal{S}^q$, I define

$$\mathcal{V}_{gl}(q) \;\sim\; T^4 \;\left(q^2(1-q)^2 \;+\; f_g\right) \;; \tag{67}$$

$f_g T^4$ is proportional to the free energy of gluons at a temperature T. As noted above, this potential is only valid in the region $0 \leq q \leq 1$; it is periodic, with period one, outside of this region:

$$\mathcal{V}_{gl}(q+1) \;=\; \mathcal{V}_{gl}(q) \;. \tag{68}$$

Obviously, $q = 0$ and $q = 1$ are degenerate,

$$\mathcal{V}_{gl}(0) \;=\; \mathcal{V}_{gl}(1) \;. \tag{69}$$

This follows from the $Z(2)$ symmetry of the pure glue theory.

The quark contribution is computed similarly:

$$\mathcal{V}_{qk}(q) \;\sim\; T^4 \;\left(2\,q^2 \;-\; q^4 \;+\; f_{qk}\right) \;; \tag{70}$$

$f_q k T^4$ is proportional to the quark contribution to the free energy. This expression is only valid for $0 \geq q \geq 1$; else q is defined modulo one. Consequently,

$$\mathcal{V}_{qk}(q+2) \; = \; \mathcal{V}_{qk}(q) \; . \tag{71}$$

This must be true for any potential, since $q = 0$ and $q = 2$ both give $\ell = +1$. Moreover, while $q = 1$ is a an extremal point of the potential, it is a local maximum, not a minimum, with

$$\mathcal{V}_{qk}(1) \; > \; \mathcal{V}_{qk}(0) \; . \tag{72}$$

This shows how quarks violate the global $Z(2)$ symmetry of the two color theory.

To one loop order, the total potential for q is the sum of the gluon and quark contributions. With many (light) quark flavors, the total potential is like the quark contribution, with a maximum at $q = 1$. Dixit and Ogilvie [38] first noticed that if the number of quark flavors isn't too large (or if the quarks are sufficiently heavy), $q = 1$ can be a local minimum; *i.e.*, $q = 1$ is *meta*stable.

The above carries through for an arbitrary number of colors and flavors. If metastable states arise, they are necessarily a $Z(N)$ state, with ℓ a (nontrivial) Nth root of unity. They are termed "$Z(N)$ bubbles" [39, 38].

This all appears to be directly analogous to the usual problem of metastable vacua, but there is one important difference. For an ordinary potential, either in quantum mechanics or in field theory, one never worries about the zero of the potential, as that can be shifted at will. In the present case, however, the zero of the potential *is* physical, and gives the free energy of the stable vacuum. This is because the "potential" is multiplied by an overall factor of T^4, and thermodynamically, derivatives with respect to the temperature matter.

Thus there is no freedom to change the zero of the potential for q. For some $Z(N)$ bubbles, if the potential at $q \neq 0$ is much higher than $q = 0$, it is well possible that the pressure in the bubble isn't positive, but *negative*! This was noticed first by Belyaev, Kogan, Semenoff, and Weiss [39].

This is a complete disaster. I suggest a possible resolution.

When we deal with q, we are in fact dealing with an *angular* variable. If we write the potential for q in terms of the thermal Wilson line, it is

$$\mathcal{V}_{pert}(q) \; \sim \; T^4 \, q^2 \, (1-q)^2 \; \sim \; T^4 \, \mathrm{tr} \left((\log \mathbf{L})^2 \left(\mathbf{1} \; - \; (\log \mathbf{L})^2 \right) \right) \; . \tag{73}$$

This form is correct in perturbation theory, where at each point in space, $\mathbf{L}(\vec{x})$ is an element of $SU(N)$. As will become clear in the next section, however, if we construct an effective theory for $\mathbf{L}$, it no longer is an element of $SU(N)$. Then this potential, for the purely angular part of $\mathbf{L}$, is ill defined when its modulus vanishes. This ambiguity is easily cured by multiplying by an overall factor of the modulus:

$$\mathcal{V}_{non-pert}(\mathbf{L}) \; \sim \; T^4 \; \left(|\ell|^2 \; + \; \ldots \right) \mathrm{tr} \; \left((\log \mathbf{L})^2 \left(\mathbf{1} \; - \; (\log \mathbf{L})^2 \right) \right) \; . \tag{74}$$

This is rank conjecture: it is certainly a *non*-perturbative modification of the potential. There is no reason to exclude terms of higher order in $\sim |\ell|^2$.

This still does not solve the problem of the zero of the potential. I now assume further that the Polyakov Loop Model (PLM) applies [26, 27, 28, 29, 30, 31]. Ignoring the angular variation in $\log \mathbf{L}$, the PLM potential is

$$\mathcal{V}_{PLM}(\ell) \;\sim\; T^4 \left(b_2 \, |\ell|^2 \;+\; b_4 \, (|\ell|^2)^2 \right) \; . \tag{75}$$

The exact potential depends upon the number of colors and flavors, *etc.*, but this is inessential here. For $q = 0$, the "usual" free energy is given by minimizing $\mathcal{V}_{PLM}(\ell)$ with respect to ℓ; with the above convention, the "mass" squared for ℓ is negative in the deconfined phase, $b_2 < 0$, and positive in the confined phase, $b_2 > 0$.

The complete potential is the sum of $\mathcal{V}_{non-pert}(\mathbf{L})$ and $\mathcal{V}_{PLM}(\ell)$. At fixed ℓ, as before any metastable points are elements of $Z(N)$. The equation which determines ℓ, however, is changed, as any metastable state has an action which acts like a *positive* mass term for ℓ. Thus the expectation value of ℓ in a $Z(N)$ bubble — even deep in the deconfined phase — has $\ell < 1$, not $\ell = 1$. Before a $Z(N)$ bubble develops negative pressure, $\ell = 0$, with zero pressure in the PLM. More likely, $Z(N)$ bubbles become unstable in the ℓ direction before $\ell = 0$.

This could be tested on the Lattice. Compute in a theory in which the splitting between the true vacuum and the metastable state is small; (dynamical) heavy quarks will do. Then the expectation value of ℓ should be smaller in the metastable state than in the stable vacuum. This holds regardless of the question of renormalization discussed in the next section.

If true, all perturbative calculations of the lifetime of a metastable $Z(N)$ bubble are wrong [38]. At best, they are a upper bound on the true lifetime, which is somewhat useless. On the other hand, it resurrects the possibility that $Z(N)$ bubbles — which spontaneously violate CP symmetry — might have appeared in the early Universe.

4 Renormalization of the Wilson Line

The sections following this deal with mean field theory for the thermal Wilson line. Implicitly, this assumes that it is possible to go from the bare Wilson line, as measured on the Lattice, to the renormalized quantity. How to do this on the Lattice is presently an unsolved problem; in this section I review what is known [40].

In a pure gauge theory, the expectation value of a closed Wilson loop, of length L and area A, is

$$\langle \mathrm{tr}\, \mathcal{P}\, \exp\left(ig \int A_\mu dx^\mu \right) \rangle \;=\; \exp\left(-m_0 L - \sigma A \right) \; . \tag{76}$$

The string tension, σ, is nonzero in the confined phase, $T \leq T_c$, and vanishes in the deconfined phase, $T > T_c$.

The concern here is not with the term proportional to the area of the loop, but with the length. This is a type of mass renormalization for an infinitely heavy quark. For example, it is easy to compute this to lowest order in perturbation theory. We are interested in an ultraviolet divergent term, so over short distances, it suffices to assume that the loop is straight. For the sake of discussion, assume that the loop runs in the time direction. (New divergences arise when there are cusps in the loop; these divergences can also be computed perturbatively, by a similar analysis [40].) Then to lowest order, there is a contribution

$$\sim \; -g^2 \, \langle \int_0^\beta \, d\tau_1 \, \int_0^\beta \, d\tau_2 \; A_0(\vec{x},\tau_1) A_0(\vec{x},\tau_2) \rangle \sim \; -\frac{g^2}{T} \, \int \, d^3k \, \frac{1}{\vec{k}^2} \; . \tag{77}$$

The integral is nominally divergent, but with either dimensional or Pauli-Villars regularization, the divergence vanishes [40]. This cancellation is somewhat trivial at one loop order, arising from having three powers of momentum upstairs, and two powers downstairs.

Thus one would expect divergences to arise at $\sim g^4$, which are found. However, for closed Wilson loops, *all* such divergences can be absorbed into charge renormalization [40]. This is a notable result: in a quantum field theory, generally the renormalization of any composite operator requires the calculation of its mixing with all other operators with the same mass dimension and symmetries. Like the action itself, however, the Wilson loop has a privileged status; it doesn't mix with any other operator.

As noted first by Polyakov [40], this result can be understood on the basis of reparametrization invariance for the Wilson loop. We parametrize the loop as a curve $x^\mu(s)$, where s is the length along the path. Then with $\dot{x} = dx^\mu/ds$, the term

$$\int \sqrt{\dot{x}^2} \; ds \tag{78}$$

is invariant under $s \to s'(s)$. Generally, physics shouldn't depend upon how we label path length along the curve. Because it has dimensions of length, however, the coefficient of this term must have dimensions of mass. With dimensional regularization, there is no such mass scale. (The renormalization mass scale only enters to ensure the proper running of the coupling constant.) A term which has no mass dimension is

$$\int \sqrt{\ddot{x}^2} \; ds \; , \tag{79}$$

$\ddot{x} = d^2x^\mu/ds^2$. This is not reparametrization invariant, though, and so does arise with dimensional regularization.

On the other hand, assume that the regularization scheme *does* introduce a mass scale. On the Lattice, this is the inverse lattice spacing, $\sim 1/a$. Then a term proportional to the length, L, does appear [40],

$$\sim \; -g^2 \, \frac{L}{a} . \tag{80}$$

At nonzero temperature, $L/a = N_t$, the number of lattice steps in the time direction. Clearly, this is the first term in an infinite series in the coupling constant, g.

How to deal with the power divergences generated by the Lattice is at present an unsolved problem. Since in the continuum there are neither logarithmic nor even finite terms to worry about, this appears to be a technical, albeit important, problem to solve.

Why is this important? In the confined phase, this constant is not of any particular consequence, as the Wilson loop is dominated by the string tension. In the high temperature phase, however, the trace of the thermal Wilson line is the order parameter for the phase transition. It would be peculiar if a precise physical definition did not exist. Any composite operator requires a condition to fix its renormalized value; for the thermal Wilson line, the natural prescription is that Polyakov loops approach one as $T \to \infty$.

To compute the leading perturbative correction to the thermal Wilson line, it is necessary to include effects from the Debye mass, $m_D \sim gT$. Replacing the bare propagator for A_0, $1/\vec{k}^2$, by $1/(\vec{k}^2 + m_D^2)$,

$$\sim \; -\frac{g^2}{T} \int d^3k \, \frac{1}{\vec{k}^2 + m_D^2} \; . \tag{81}$$

This divergent integral can be computed using either dimensional or Pauli-Villars regularization. Or, one can just subtract the integral with $m_D = 0$:

$$-\frac{g^2}{T} \int d^3k \, \left(\frac{1}{\vec{k}^2 + m_D^2} - \frac{1}{\vec{k}^2} \right) \sim -\frac{g^2}{T} \, (-m_D) \; \sim \; + \, g^3 \; . \tag{82}$$

This was first demonstrated by Gava and Jengo [41]. That is, while the leading term is negative in the bare theory, it is positive after regularization. This change in sign is unremarkable, as the sign of a renormalized operator is not preserved under regularization.

This appears to indicate that the renormalized thermal Wilson line is not a unimodular matrix. One concern is that any quantity $\sim g^3$ really arises from a two-loop graph, $\sim g^4$, times an infrared singular piece $\sim 1/m_D \sim 1/g$. Thus it is necessary to ensure that the above is the only infrared singular term at this order.

A different calculation was performed by Korthals Altes [33]. He computed the one loop corrections to the thermal Wilson line in a background A_0 field.

The method is identical to that used in sec. II to compute the interface tension. Classically, the thermal Wilson line is a special unitary matrix. Korthals Altes finds that the one-loop corrections to the thermal Wilson line are not only infinite (!), but generate a matrix which is neither unitary nor special. On the other hand, all Polyakov loops are finite.

Thus even in the continuum, the renormalization of the thermal Wilson line, and Polyakov loops, remains an unsolved problem.

A way of measuring renormalized Polyakov loops on the Lattice has been proposed by Zantow *et al.* [43]. They compute only two-point functions of the Polyakov loop. At short distances, the static potential can be computed perturbatively, which allows one to extract the renormalized Polyakov loop.

5 Deconfining Transition for Two, Three, and Four Colors

5.1 Polyakov Loops and Quarkless Baryons

So far I have been concerned with the (pure glue) theory at very high temperatures. Now I turn to the question of its behavior near the critical temperature. I review Lattice results on the order of the phase transition for two [20], three [21], and four [17] colors, and then ask what constraints it places on the mean field theory for Polyakov loops.

Up to this point, I have only considered the trace of the thermal Wilson line in the fundamental represenation, which is the the Polyakov loop ℓ. By a local gauge transformation, at each point in space one can diagonalize the thermal Wilson line. These eigenvalues are gauge invariant, so since $\mathbf{L}(\vec{x})$ is an $SU(N)$ matrix, at each point there are $N-1$ independent degrees of freedom. Another $N-1$ degrees of freedom are given by the trace of powers of $\mathbf{L}$, $\mathrm{tr}\mathbf{L}^j$, $j = 1 \ldots (N-1)$.

Under a global $Z(N)$ transformation, the "usual" Polyakov loop transforms as a field with charge-one, eq. (14); thus I relabel it ℓ_1. Polyakov loops with higher $Z(N)$ charge are easy to construct. I introduce the traceless part of $\mathbf{L}$:

$$\widetilde{\mathbf{L}} = \mathbf{L} - \ell_1 \mathbf{1} \; . \tag{83}$$

Then I define the charge-two Polyakov loop to be

$$\ell_2 = \frac{1}{N}\,\mathrm{tr}\,\widetilde{\mathbf{L}}^2 \;=\; \frac{1}{N}\,\mathrm{tr}\,\mathbf{L}^2 \;-\; \frac{1}{N^2}\,\left(\mathrm{tr}\,\mathbf{L}\right)^2 \;, \tag{84}$$

where

$$\ell_2 \to e^{2i\phi}\ell_2 \; , \tag{85}$$

with ϕ as in eq. (6). There are two operators with charge-two, ℓ_2 and ℓ_1^2.

For example, consider two colors, and the parametrization of the thermal Wilson line in the strict perturbative regime, (24). The charge-one loop is $\ell_1 = \cos(\pi q)$, eq. (25), while the charge-two loop is $\ell_2 = -\sin^2(\pi q)$. At high temperature, where $q = 0, 1$, $\langle \ell_1 \rangle \to \pm 1$, while $\langle \ell_2 \rangle \to 0$.

I note that Polyakov loops of charge-two and beyond are related to the trace of the thermal Wilson line in higher $SU(N)$ representations. For two colors, in perturbation theory the trace of the Wilson line in the adjoint representation is $\mathrm{tr}(\mathbf{L}_{adj}) = 1 + 2\ell_2$. So far, though, I haven't found this particularly useful.

Continuing on,

$$\ell_3 = \frac{1}{N}\,\mathrm{tr}\,\widetilde{\mathbf{L}}^3 \tag{86}$$

has charge three under the global $Z(N)$ symmetry. Other charge-three operators are ℓ_1^3 and $\ell_1\ell_2$. The construction of operators with higher $Z(N)$ charge proceeds similarly. For example, operators with charge four, independent of the singlet part, are given by $\mathrm{tr}\,\widetilde{\mathbf{L}}^4$ and $(\mathrm{tr}\,\widetilde{\mathbf{L}}^2)^2$, *etc.*

I stress that both the expectation values, and correlations functions of, Polyakov loops of arbitrary charge are well worth measuring on the Lattice. When the $Z(N)$ symmetry is spontaneously broken at $T > T_c$, Polyakov loops ℓ_j with charge $j = 1 \ldots (N-1)$ all acquire nonzero expectation values. As Polyakov loops of charge-two and beyond are constructed from the traceless part of the thermal Wilson line, they aren't that interesting at high temperature; as $T \to \infty$, their expectation values are proportional to nonzero powers of g^2, times powers of T to make up the mass dimension. Near T_c, however, there is nothing general which can be said about their expectation values.

I will assume that for a second order deconfining transition, the only critical field is the charge-one loop [3]. Even so, Polyakov loops of charge-two and beyond will certainly affect non-universal behavior. One notable example is the Polyakov Loops Model [25, 26, 27, 28, 29, 30, 31], which conjectures a relationship between the expectation value of Polyakov loops and the pressure. The original model assumed that only the charge-one loop mattered, but I no longer see why the expectation values of higher-charge loops are not important as well.

There are certain Polyakov loops which have a privileged status: these are those with charge-N. As they are neutral under $Z(N)$, their expectation values are nonzero at any temperature. I term such operators *quarkless baryons.*

In QCD, a baryon is N quarks tied together through an antisymmetric tensor in color space. One can also consider a more general object, a baryon "junction" [49]. This is an antisymmetric color tensor, with N Wilson lines coming out of it. Putting quarks at the end of each line gives the usual QCD baryon, since in the confined phase, Wilson lines are short, on the order of $\sim 1/\sqrt{\sigma}$, where σ is the string tension. While directly related to QCD baryons, without quarks, baryon junctions are not gauge invariant: only junction anti-junction pairs are.

In contrast, all quarkless baryons are gauge invariant. The simplest quarkless baryon is ℓ_1^N. In mean field theory, this is zero in the confined phase, and nonzero above. In the full quantum theory, $\langle \ell_1^N \rangle \neq 0$ at all T. While there is not good Lattice data on this expectation value, I assume that it is small below T_c, and large above, but this is just a guess. Since junction anti-junction pairs involve N Wilson lines, they are directly related to the operators for quarkless baryons, ℓ_1^N, *etc.*.

5.2 Effective Theories for Polyakov Loops

I next turn to the construction of an effective theory for the thermal Wilson line. Remember how this proceeds with an Ising model on a lattice. While the value of the spin on each site is ± 1, after an effective spin is computed by averaging over a domain of fixed size, the result effective spin is a continuous variable, $\phi(\vec{x})$. The effective theory is just the usual ϕ^4 theory. By the renormalization group, it is in the same universality class as the original Ising model.

The analogous proceedure can be carried through for the thermal Wilson line. The effective thermal Wilson line, constructed by a gauge invariant [25] average over a domain of some size, is not an $SU(N)$ matrix, but has more degrees of freedom. I then consider *all* Polyakov loops, from charge-one up to charge-N. Of course there is no reason to stop there, but presumably the number of effective fields which really matters is limited.

In an effective Lagrangian, the first thing to ask about are the mass terms:

$$\mathcal{L}^{eff} \; = \; m_1^2 |\ell_1|^2 \; + \; m_2^2 |\ell_2|^2 \; + \ldots \tag{87}$$

The simplest assumption is that for $T \geq T_c$, condensation is driven by a negative mass term for the charge-one loop:

$$m_1^2 < 0 \quad , \quad T \, > \, T_c \quad , \quad m_1^2 > 0 \quad , \quad T \, < \, T_c \; , \tag{88}$$

and that the masses for all higher loops are positive at all temperatures,

$$m_2^2 > 0 \quad , \quad m_3^2 > 0 \ldots \tag{89}$$

If so, then the charge-one loop controls the critical behavior [3].

There is good reason why one expects that condensation is driven by that of the charge-one loop, and *not* by loops with higher charge. If the mass for the charge-one loop is negative, the favored vacuum is given by maximizing $|\mathrm{tr}\mathbf{L}|^2$. After a global gauge rotation, we can always choose the expectation value of $\mathbf{L}$ to be a diagonal matrix. If $\mathbf{L}$ were a $U(N)$, instead of an $SU(N)$ matrix, then $|\mathrm{tr}\mathbf{L}|^2$ is maximized when $\mathbf{L}$ is a constant phase times the unit matrix. This remains true when $\mathbf{L}$ is a $SU(N)$ matrix; for it to be a unit matrix, however, it must be an element of the center of the gauge group,

$$\langle \mathbf{L} \rangle \; = \; \ell_0 \; \exp{(i\phi)} \; \mathbf{1} \; . \tag{90}$$

with ϕ a $Z(N)$ phase, $\phi = 2\pi j/N$, $j = 1 \ldots (N-1)$.

For this particular expectation value, the vacuum does *not* spontaneously break the (global) $SU(N)$ symmetry above T_c. This accords with naive expectation: the high temperature vacuum is not in a Higgs phase. The possibility of having an expectation value which doesn't break $SU(N)$ is special to a field in the adjoint, as opposed to the fundamental, representation.

On the other hand, assume that $m_1^2 > 0$, and $m_2^2 < 0$ in the deconfined phase, so that symmetry breaking is driven by condensation of the charge-two, instead of the charge-one, loop. Then the vacuum is given by maximizing $|\mathrm{tr}\mathbf{L}^2|^2$; this means that the expectation value of $\mathbf{L^2}$ is an element of the center. But if so, besides the $SU(N)$ invariant vacuum, there are also vacua which are only invariant under $SU(N-1)$. At present, there is no evidence to suggest that the high temperature vacuum is one where $SU(N)$ spontaneously breaks to $SU(N-1)$.

The above description can be extended beyond mean field theory, at least if the deconfining transition is of second order. A transition driven by the charge-one loop is one where $m_1^2 \to 0$ at T_c; one driven by the charge-two loop is where $m_2^2 \to 0$ at T_c, *etc.* To be precise, for a transition driven by a charge-k loop, both its mass, and that of the charge-$(N-k)$ loop, vanish at T_c.

The masses for Polyakov loops of all charges can be directly measured on the Lattice. Even for the charge-one loop, data near T_c is, at present, limited [8, 22]. There is also some data for higher charge loops for three colors in $2+1$ dimensions [51].

Given this (crucial!) assumption about the masses, I next turn to the order of the phase transition for a small number of colors.

5.3 Two Colors: Second Order, and Quarkless Baryons

For two colors, all Polyakov loops are real. For the charge-one loop, I take the potential

$$\mathcal{V}_1 \;=\; \frac{m_1^2}{2}\,\ell_1^2 \;+\; \frac{\lambda_1}{4}\,\ell_1^4 \,, \tag{91}$$

with a positive quartic coupling, $\lambda_1 > 0$. Of course higher powers in ℓ_1 are also possible. Near a second order phase transition, however, the most relevant operators, with the fewest powers of ℓ_1, dominate.

Invariant terms involving the charge-two loop include

$$\mathcal{V}_2 \;=\; h\,\ell_2 \;+\; \frac{1}{2}\,m_2^2\,\ell_2^2 \;+\; \ldots \tag{92}$$

plus terms $\sim \ell_2^3$, $\sim \ell_2^4$, *etc.* All powers of ℓ_2 are allowed because it is a singlet. The potential which mixes the charge-one and charge-two loops starts out as

$$\mathcal{V}_{mix} \;=\; \xi\,\ell_1^2\,\ell_2 + \ldots \,. \tag{93}$$

plus many other terms; this has the lowest mass dimension.

There is extensive Lattice data on the nature of the deconfining phase transition [20]. Especially from the work of Engels *et al*, it appears that the transition is of second order. To wit, the critical exponents are within $\sim 1\%$ of the values expected for the Ising model [3].

There is a surprise, however. As first stressed by Damgaard [20], the expectation value of the Polyakov loop in the adjoint representation is an approximate order parameter. In perturbation theory, the adjoint Polyakov loop is $1 + 2\ell_2$, so from the Lattice data, the expectation value of ℓ_2 presumably jumps at T_c.

From the terms above, the expectation value of the charge-two loop is

$$\langle \ell_2 \rangle \; = \; - \, \frac{h + \xi^2 \, \ell_0^2}{m_2^2} \; . \tag{94}$$

To explain the jump in $\langle \ell_2 \rangle$ about T_c, there are then two possibilities. If the charge-two loop is heavy, then the coupling constant of the charge-two loop to the charge-one loop, ξ, must be large. This means that changes in the density of the charge-two loop is driven by condensation of the charge-one loop.

The other possibility is that h and ξ are not especially large, but that the charge-two loop becomes light near T_c. The latter doesn't violate universality, as long as the charge-two loop isn't massless at T_c.

There is no lattice data on $\langle \ell_1^2 \rangle$. I presume that as suggested by mean field theory, ℓ_1^2 quarkless baryons are rare below T_c, and common above. This is seperate from the changes in ℓ_2.

5.4 Three Colors

5.4.1 A "Nearly" Second Order Transition

In an asymptotically free gauge theory, it is natural to form the ratio of the true pressure to that of an ideal gas. In principle, positivity of entropy does not require this ratio to be less than one. In practice, Lattice data with improved actions finds that this ratio is less than one at all temperatures [5].

Numerical simulations find that for three colors, the deconfining transition is of first order [21], in agreement with general arguments [3]. For a first order transition, the pressure is continuous at T_c, but the energy density jumps. Thus consider the ratio of the jump in the energy density to that of an ideal gas.

This ratio is not bounded by one. To illustrate this, consider a bag model. Above T_c, the pressure is that of an ideal gas, minus a bag constant, b:

$$p_{bag} \; = \; c_0 T^4 \; - b \; . \tag{95}$$

The pressure is assumed to vanish below T_c, with T_c fixed by $p_{bag} = 0$. This bag model does not describe the Lattice data near T_c, but is a useful construct. In the bag model, the ratio of energies is:

$$\frac{\delta e}{e_{ideal}} |_{Bag} \; = \; \frac{4}{3} \; . \tag{96}$$

In contrast, Lattice data appears to give a result which is much smaller:

$$\frac{\delta e}{e_{ideal}}|_{Lattice} \sim \frac{1}{3} \, . \tag{97}$$

The mass of the charge-one loop has also been measured on the Lattice [22]. It goes from $m_1/T \sim 2.5$ at $\sim 2T_c$, down to $m_1/T \sim .25$ at $\sim T_c^+$. This decrease in the screening mass, apparently by a factor of ten, strongly suggests the in fact the transition is even weaker than the above comparison with the bag model suggests. Instead of weakly first order, I prefer to call the deconfining transition *nearly second order*.

(It is necessary to measure correlation functions of Polyakov loops to see this decrease. Masses measured from other operators, such as plaquettes, do not decrease dramatically about T_c [8]. This implies that the mixing between Polyakov loops and and plaquette operators are small. This small mixing is found in related problems [7].)

An effective theory cannot explain why the deconfining transition is weakly first order; it merely requires that certain coupling constants are small.

5.4.2 Polyakov Loops with Charge One and Minus One

For three colors, I consider Polyakov loops with charge one, two and three.

There is no data on the expectation values for the quarkless baryons, ℓ_1^3, $\ell_1 \ell_2$, and ℓ_3. I presume that as indicated by mean field theory, $\langle \ell_1^3 \rangle$ is small below T_c, and large above. It would be interesting to knowhow the density of the other quarkless baryons, $\ell_1 \ell_2$ and ℓ_3, change about T_c. Regardless of the renormalization issues discussed in sec. (4), changes in these expectation values are presumably physical.

Thus I concentrate on the interaction between the charge-one and the charge-two loops. Remember that for three (or more) colors, the ℓ_j's are all complex valued fields. The potential for charge-one loops is dictated by the global $Z(3)$ symmetry:

$$\mathcal{V}_1 \; = \; m_1^2 \, |\ell_1|^2 \; + \; \kappa_1 \; \left(\ell_1^3 \; + \; (\ell_1^*)^3 \right) \; + \; \lambda_1 \; \left(|\ell_1|^2 \right)^2 \; . \tag{98}$$

The notable feature is the appearance of a cubic term, which necessarily ensures a first order transition [3].

The charge-two loop has charge minus one under $Z(3)$, so its potential is the same, albeit with different masses and coupling constants:

$$\mathcal{V}_2 \; = \; m_2^2 \, |\ell_2|^2 \; + \; \kappa_2 \; \left(\ell_2^3 \; + \; (\ell_2^*)^3 \right) \; + \; \lambda_2 \; \left(|\ell_2|^2 \right)^2 \; . \tag{99}$$

There are many terms by which the charge-one and charge-two fields can mix. The most important is that with the smallest mass dimension:

$$\mathcal{V}_{mix} \; = \; \xi \, (\ell_1 \ell_2 \; + \; \ell_1^* \ell_2^*) \; . \tag{100}$$

In terms of the original thermal Wilson line, this term is $\sim (\mathrm{tr}\mathbf{L})(\mathrm{tr}\mathbf{L}^2)$, *etc.*

If the charge-two field is heavy near T_c, we can integrate it out. While it may be a mess to do so analytically, any resulting potential, involving only ℓ_1, must still respect the overall $Z(3)$ symmetry. This produces a potential identical in form to $\mathcal{V}_1$, but with different values for the mass and coupling constants. A weakly first order requires that the effective cubic coupling in the resulting effective theory is small, $\widetilde{\kappa}_1 \ll 1$.

If the charge-two loop becomes light near T_c, and if it mixes strongly with the charge-one loop through a large coupling constant ξ, then its effects cannot be neglected.

There is another possibility. The mass and quartic terms in the potentials are invariant not just under $Z(3)$, but under a global $U(1)$ symmetry. Assume that the charge-two field is always heavy. Then all terms in both potentials are invariant under a global $U(1)$, with the exception of the cubic terms, with couplings κ_1 and κ_2, *and* the mixing term between ℓ_1 and ℓ_2, with coupling ξ. Thus perhaps *all* terms invariant under $Z(3)$, but not $U(1)$, are small. That is, the heavy charge-two loop mixes weakly with the charge-one loop. This doesn't explain why all $Z(3)$ couplings are small, but hints at a more general principle.

It will be interesting to see what detailed numerical studies on the Lattice tell us.

5.5 Four Colors: First Order from Charge-Two Loops

For four colors, I consider just the charge-one and charge-two loops. Under $Z(4)$, $\ell_1 \to i\ell_1$, so the potential for the charge-one field alone is

$$\mathcal{V}_1 \;=\; \frac{m_1^2}{2}\,|\ell_1|^2 \;+\; \lambda_1\,(|\ell_1|^2)^2 \;+\; \kappa_1\,\left(\ell_1^4 \;+\; (\ell_1^*)^4\right)\;. \tag{101}$$

The term $\sim \lambda_1$ is $O(2)$ invariant, while that $\sim \kappa_1$ is only invariant under $Z(4)$.

Under $Z(4)$, $\ell_2 \to -\ell_2$, so the potential for the charge-two field by itself is just like that for the charge-one field:

$$\mathcal{V}_2 \;=\; +\,m_2^2\,|\ell_2|^2 \;+\; \lambda_2\,\left(|\ell_2|^2\right)^2 \;+\; \kappa_2\,\left(\ell_2^4 \;+\; (\ell_2^*)^4\right)\;. \tag{102}$$

The allowed terms which mix the two fields are:

$$\mathcal{V}_{mix} \;=\; \zeta_1\,\left(\ell_2^*\,\ell_1^2 \;+\; \ell_2\,(\ell_1^*)^2\right) \;+\; \zeta_2\,\left(\ell_2\,\ell_1^2 + \ell_2^*\,(\ell_1^*)^2\right)\;. \tag{103}$$

Unlike two colors, a term linear in ℓ_2 is not allowed by the $Z(4)$ symmetry. For $\zeta_2 = 0$, and assuming that the charge-two field reamins heavy about T_c, ℓ_2 can be integrated out to give:

$$\sim\; -\,\frac{\zeta_1^2}{m_2^2}\,(|\ell_1|^2)^2\;. \tag{104}$$

One can convince oneself that this term is necessarily negative. That is, the quartic coupling for the charge-one field is shifted downward:

$$\widetilde{\lambda}_1 \;\equiv\; \lambda_1 - \frac{\zeta_1^2}{m_2^2} \; . \tag{105}$$

The same holds if both ζ_1 and ζ_2 are nonzero: integrating out the charge-two field generates corrections to the quartic coupling constants of the charge-one field which are uniformly negative, shifting them downwards.

The transition for four colors appears to be of first order [17]. One explanation for this is that the original coupling constants λ_1 and λ_2 are positive, but after integrating out the charge-two field, they become negative, and thus drive the transition first order.

The analysis for higher numbers of colors is then immediate. The leading term which couples a charge-j loop to the charge-one loop is

$$\mathcal{V}_{mix} \;=\; \xi \left(\ell_j^* \, \ell_1^j \;+\; \ell_j \, (\ell_1^*)^j \right) \; . \tag{106}$$

Assuming that the charge-j loop is heavy about T_c, we can integrate it out, which produces a term in the potential for the charge-one loop $\sim (|\ell_1|^2)^j$. For charges greater than two, this is less relevant (has smaller mass dimension) than the quartic terms expected to dominate.

Thus if the deconfining transition is of first order for more than four colors, in the present language it is uniquely due to how the coupling between charge-two and charge-one loops affects the effective quartic coupling constant for the charge-one loop. Polyakov loops with charge greater than two do not affect the order of the transition.

6 A Parting Comment

While it is true that, in equilibrium, all thermodynamic quantities follow from the pressure, experience with the perturbative calculation of processes near equilibrium — such as transport coefficients, real photon production, *etc.* — teaches us they often depend on the details of equilibrium correlation functions. Thus regardless of theoretical prejudice, such as the Polyakov Loops Model, it is important to measure as many gauge invariant correlation functions as possible.

There is an astounding amount of superb data which is pouring out of RHIC [45]. Many features, including the change in the spectrum of “hard” particles, details of Hanbury-Brown-Twiss interferometry, chemical composition, *etc.*, appear to defy explanation by any conventional mechanisms [52]. We may very well need a detailed understanding of the theory, near T_c and above, in order to sort out these amazing results.

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